Analysis of inventory systems for non-perishable goods with changing production rates and random switching times.

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ABSTRACT

This paper investigates a stochastic inventory model for a single product operating under two distinct production rates, with demand assumed to follow a Poisson distribution. The items considered are non-perishable over the planning horizon. When the inventory reaches a specified threshold, the system transitions from an OFF state to an ON state, subject to a switching delay characterized by an exponential distribution with parameter τ. During this switching period, no demand is fulfilled, and all arriving demand is lost. The model incorporates backlogging, where the production rate under backlog conditions is higher than that of regular production. Time-dependent system characteristics are analyzed and presented to highlight the transient behavior of the inventory process.

Keywords: *Product inventory systems, stochastic modeling, Poisson demand distribution, non-perishable items, switching time analysis, backlog policies, level-dependent production, exponential switching behavior.*
AMS Subject Classifications: 90B05, 90B30.

1.0 INTRODUCTION

Inventory control represents a fundamental component of supply chain management. Traditional models often assume that products do not deteriorate within the period under consideration. However, in practical scenarios, a certain degree of deterioration or wastage is unavoidable. Consequently, analysts and decision-makers frequently require information on system behavior over a finite horizon, prior to the attainment of steady-state conditions. In such contexts, transient analysis becomes indispensable. It is widely applied in manufacturing environments, systems subject to failures, unstable queuing networks, and workloads exhibiting variability or non-stationarity. Transient analysis further enables the evaluation of performance within limited intervals, sensitivity studies, first-passage time and settling time computations, and projections of model behavior as it converges toward equilibrium. Transient analysis (finite-horizon) reveals behavior not visible from steady-state metrics; for systems with significant switching/setup time, transient performance over practical horizons may dominate long-run averages. The transient study provides methods to compute time-dependent distributions and first-passage quantities.

2.0 LITERATURE REVIEW

The study of production–inventory systems with multiple production rates and random switching or setup times have gained increasing attention in recent years. Traditional inventory models often assume constant production rates and negligible setup delays, focusing primarily on steady-state analysis. However, such assumptions fail to capture the operational realities of modern production environments, where switching between different production rates involves stochastic setup times and demand is highly variable. A more recent stream of research directly addresses transient analysis in two-rate production systems under Poisson demand. In these models, switching or setup times are frequently assumed to follow exponential distributions, and demand during switching periods may result in lost sales. A particularly relevant contribution is the paper *Transient Analysis of Production Inventory System with Different Rates of Production and Random Switching Time (2019)*, which extends beyond steady-state performance measures by developing exact transient solutions. These include distributional and transform-based results for inventory levels up to finite time horizons. The literature highlights that random switching and production-rate changes significantly increase the likelihood of lost sales or backorders, especially when switching occurs near low inventory thresholds. Furthermore, the distinction between lost-sales and backorder assumptions is shown to be pivotal for determining optimal control strategies. To alleviate shortage risks, state-dependent production policies – for example, switching to a higher production rate when inventory falls below a given threshold – have been proposed. While these approaches help reduce shortages with limited increases in production cost, they also introduce analytical complexity, often necessitating numerical or simulation-based methods.

Importantly, scholars emphasize that transient analysis provides insights into short-term system dynamics that are not observable from steady-state models. In systems where switching/setup times are non-negligible, finite-horizon performance measures may be more operationally relevant than long-run averages. The 2020/2021 study in particular presents a rigorous methodology for computing time-dependent distributions and first-passage probabilities, thereby offering a comprehensive framework to understand the short-term behavior of stochastic production–inventory models with non-perishable goods. Ben-Daya et al. (2008) investigated an Economic Production Quantity (EPQ) model variant in which production begins at one rate and subsequently shifts to another (often lower) rate after a random time. Their work develops an analytical framework for determining optimal cycle lengths and assessing cost implications when production deterioration or random rate shifts occur. This line of research directly contributes to the modeling of stochastic shifts in production rates within inventory systems.

Mohebbi et al. (2006), along with subsequent extensions, examined production–inventory systems with random production changes, such as failures or setup interruptions, under compound Poisson demand and finite capacity constraints. These studies focus on long-run performance, deriving cost and service-level measures while often incorporating the assumption of lost sales during switching or repair periods. Such models are particularly relevant for settings where interruptions in production are frequent and capacity is limited. Krishnamoorthy et al. (2013) and related contributions analyzed (s,S)-type and production–inventory models in which processing times or production rates vary with the current inventory position. By employing matrix-analytic and birth–death process approaches, they obtained performance measures for these state-dependent systems. This family of models is important in contexts where production speed is deliberately adjusted in response to inventory levels, thus enabling dynamic, inventory-sensitive control policies.

Recent contributions have advanced the study of transient production–inventory systems with variable production rates and random switching/setup times. A seminal work in this domain is *Transient Analysis of Production Inventory System with Different Rates of Production and Random Switching Time (2019)*, which develops a rigorous analytical framework for a single-product, two-rate production model subject to Poisson demand and exponentially distributed switching times. This study provides distributional and transform-based expressions for the inventory level over finite horizons and explicitly incorporates the effect of lost demand during switching periods, establishing a foundational reference for transient two-rate production analysis.

A subsequent study, *Time-Dependent Production Inventory Model with Random* (2022, Ataullah), extends and operationalizes this framework. This work emphasizes numerical evaluation of the transient model and introduces certain generalizations, thereby supplying practical computational methods and implementation details. It highlights the feasibility of applying transient analysis to real-world settings where explicit numerical solutions are required. In a related but broader context, Poormoaied et al. (2021) investigate inventory and ordering policies under an ON/OFF supplier model, where supply interruptions follow exponential distributions. Using continuous-time Markov chain (CTMC) methods, they derive time-dependent performance measures and propose control policies suitable for environments with random disruptions. While not strictly a two-rate production model, their results are conceptually aligned with transient inventory analysis under stochastic switching and demonstrate the wider applicability of such methods.

Further, Wang et al. (2024) apply transient methods to examine production–performance and energy trade-offs in manufacturing systems. Their study illustrates that short-run performance metrics, obtained through transient analysis, can inform operational decisions balancing throughput efficiency and energy consumption. This highlights the growing recognition of transient analysis as a practical decision-support tool in modern production environments. Finally, very recent theoretical contributions (2024–2025) point toward emerging directions in the field. These include the use of phase-type (PH) distributions to approximate non-exponential switching behavior, the development of diffusion and fluid limits for scalable transient approximations, and exploratory models that exploit stochastic lead times or disruptions for adaptive policy design. Although still in early stages, these studies suggest promising extensions of transient inventory modeling, offering tools to address more complex and realistic production systems. Taken together, this body of work demonstrates an evolution from foundational transient frameworks (2020) to computational extensions (2022), applied models under disruption (2021), practical performance trade-offs (2024), and emerging theoretical innovations (2024–2025). Collectively, these contributions underscore the importance of transient analysis for understanding and optimizing production–inventory systems with stochastic switching dynamics.

3.0 ASSUMPTIONS AND NOTATIONS

3.1 Assumptions

- The system operates with two production rates, where the production rate during backlog periods is higher than during normal operation.
- When the inventory level reaches a predetermined threshold level, denoted by *-N,* the system switches from OFF mode to ON mode, with a switching time governed by an exponential distribution.
- Perishability of items is considered to be time-dependent.
- During the switching period, no demand is fulfilled; hence, any demand occurring during this time is considered lost forever.
- When the inventory level reaches a predefined order level, the production process is switched off.

3.2 Notations

i. $\xi \rightarrow$Arrival rate,

ii. $\mu \rightarrow$ Production rate during backlogs,

iii. $\kappa \rightarrow$Normal production rate,

iv. S $\rightarrow$Maximum inventory level,

v. $\tau \rightarrow$Switching time

vi. $I(t)$ Inventory level at time t,

vii. $E \rightarrow E_1 \cup E_2$ is the state space of the process

where, $E_1 = \{(t,0): t = -N+1, \dots, S\}$ and $E_2 = \{(t,1): t = -N+1, \dots, S-1\}$

4.0 MODEL ANALYSIS

The proposed model begins with the inventory level at its maximum M and the production system in the OFF mode. Demand arrives according to a Poisson process with rate λ. Inventory depletes due to customer demand and is assumed non-perishable over time, though it is level dependent.

When the inventory level falls to the threshold $-N$, the system switches to ON mode. n the inventory system, inventory level $I(t)$ takes the value

$$A = \{-N, -N+1, \dots, 0, 1, 2, \dots, S\}$$

To get a two-dimensional Markov process we introduce the $\{X(t), t \geq 0\}$

Where, *X(t)* is defined by

$$X(t) = \begin{cases} 1 & \text{when production is ON} \\ 0 & \text{whwn production is OFF} \end{cases}$$

Now, the infinitesimal generator of the two-dimensional Markov process$\{I(t), X(t); t \geq 0\}$ is defined on the state space E. it is noted that the Markov process is a pure birth and death process during the transition from the state $(S,0)$ through the state $(S-1,0),\ldots,(-N+1,0)$ when the production process is in OFF mode. When inventory level in the state $(-N,0)$ then the system is switched ON. Switching time follows exponential distribution with parameter α and reached the state $(N,1)$from $(-N,0)$. From this state onward the process till it reaches the level $(S,0)$.

Let us assumed $I(0)=S$and $X(0)=0$. Let us consider the transition probabilities:

$$P_{(S,0)(i,j)}(t)=P\{I(t),X(t)=(i,j)|I(0),X(0)=(S,0)\}$$

From now onwards we can write

$$P_{(i,j)}(t)\text{for}P_{(S,0)(i,j)}(t)$$

Kolmogorov difference differential equations for the system $P_{(i,j)}(t)$ are given bellow:

When system is OFF mode:

$$P'_{(S,0)}(t)=-(\xi+S)P_{(S,0)}(t)+\kappa P_{(S-1,1)}(t) \quad \ldots \quad \text{(i)}$$

$$P'_{(i,0)}(t)=-(\xi+i)P_{(i,0)}(t)+\big(\xi+(i+1)\big)P_{(i+1,0)}(t);\ i=S-1,\ldots,0 \quad \ldots \quad \text{(ii)}$$

$$P'_{(i,0)}(t)=-\xi P_{(i,0)}(t)+\xi P_{(i+1,0)}(t)\ ;\ i=-1,\ \ldots\ ,-N+1 \quad \ldots \quad \text{(iii)}$$

$$P'_{(-N,0)}(t)=-\tau P_{(-N,0)}(t)+\ \xi P_{(-N+1,0)}(t) \quad \ldots \quad \text{(iv)}$$

When the system is ON mode:

$$P'_{(S-1,1)}(t)=-(\xi+i+\kappa)P_{(S-1,1)}(t)+(\xi+i)P_{(S,0)}(t) \quad \ldots \quad \text{(v)}$$

$$P'_{(i,1)}(t)=-(\xi+i+\kappa)P_{(i,1)}(t)+\big(\xi+(i+1)\big)P_{(i+1,0)}(t)+\kappa P_{(i+1,1)}(t);\ \text{i=S-2, …, 0} \quad \ldots \quad \text{(vi)}$$

$$P'_{(i,1)}(t)=-(\xi+\mu)P_{(i,1)}(t)+(\xi+\mu)P_{(i+1,1)}(t);\quad \text{i=-N+1, …,-2,-1} \quad \ldots \quad \text{(vii)}$$

$$P'_{(-N,1)}(t)=-\mu P_{(-N,1)}(t)+\tau P_{(-N,0)}(t)+\xi P_{(-N+1,1)}(t) \quad \ldots \quad \text{(viii)}$$

The system of ordinary differential equations (ODEs) is solved using the fourth-order Runge–Kutta method, employing specified parameter values. The resulting solutions are utilized to generate graphical representations of the system dynamics and associated performance measures. The study investigates the impact of varying parameter values on key performance indicators, including the expected number of customers in the system and the mean waiting time. Computational implementation and analysis are carried out using MATLAB R2016a software to ensure accuracy and efficiency.

4.1 Some performance measures

a) *Mean inventory level in the system*

Let the expected inventory level

$$Ls(t)=\sum_{i=1}^{S} iP_{(i,0)}(t)+\sum_{i=1}^{S-1} iP_{(i,1)}(t)$$

b) Expected number of perishable items

Let the number of perishable items

$$Lp(t) = \sum_{i=1}^{S} iP_{(i,0)}(t) + \sum_{i=1}^{S-1} iP_{(i,1)}(t)$$

c) Expected backlogs in the system

Let the expected backlogs

$$Lb(t) = \sum_{i=-N}^{-1} |i|P_{(i,0)}(t) + \sum_{i=-N}^{-1} |i|P_{(i,1)}(t)$$

d) Expected number of customers lost

Let expected number of customers lost $CL(t) = \xi P_{(-N,o)}(t) + \xi P_{(-N,1)}(t) + \tau\xi P_{(-N,0)}(t)$

e) Expected total cost of the system

$$\boldsymbol{ETC(t)) = L + c_1 Ls(t) + c_2 Lp(t) + c_3 Nb(t) + c_4 CL(t)}.$$

5.0 NUMERICAL RESULTS AND DISCUSSION

In all numerical computations, the model parameters are taken as

S=3, *N*=2, ξ=1, $\alpha = 0.21$, $\mu = 3$, $\kappa = 2$, L=25, $c_1 = 0.15, c_2 = 0.25, c_3 = 0.35$.

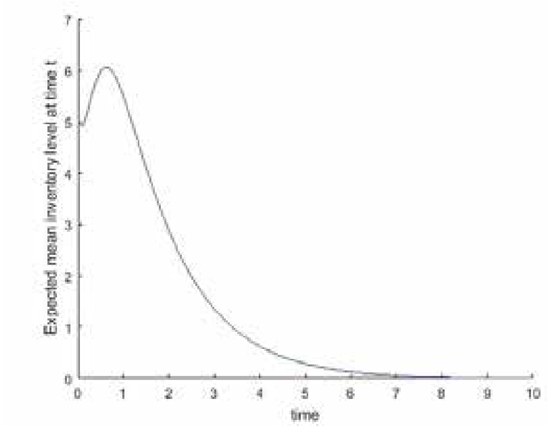

Fig.1: Mean inventory level vs Time

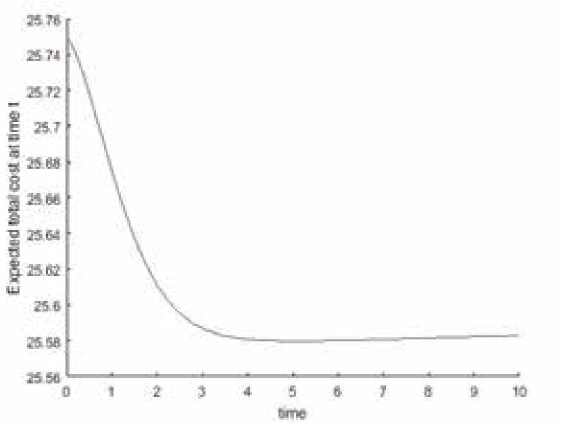

Fig.2: Expected total cost vs Time

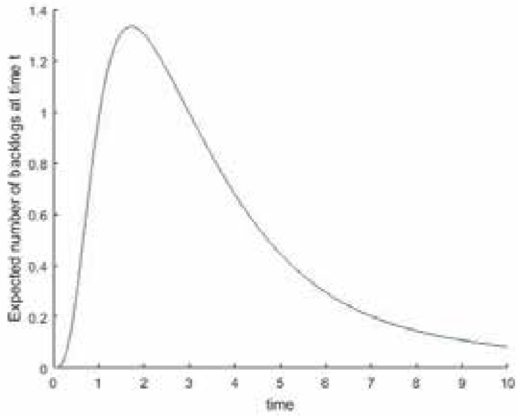

Fig.3: Expected backlogs vs Time

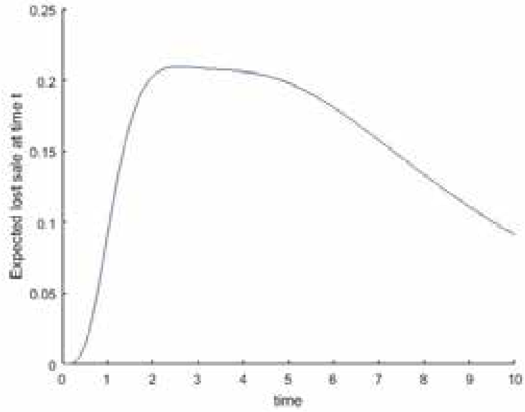

Fig.4: Expected lost sale vs Time

Numerical results were obtained by applying the fourth-order Runge–Kutta method to the system of ordinary differential equations (I) through (VIII), implemented using MATLAB R2016a. The analysis was conducted for the parametric values over the time interval $0 \leq t \leq 100$, with the results illustrated graphically. Figures 1–5 depict the behavior of different parameters as functions of time. Figure 1 shows that the inventory level increases rapidly during the initial period but decreases at a slower rate over longer time intervals. Figure 2 presents the total cost of the system, revealing that it is initially higher compared to later stages of the considered time horizon. Figure 3 illustrates the average number of backlogs, which rise sharply at the initial stage but gradually decline over time at a certain rate. Figure 4 depicts the lost-sales rate, which is initially high but decreases over time as the production rate surpasses the normal production rate. These results highlight the dynamic nature of the system and demonstrate how time-dependent variations in production and demand influence key performance measures.

6.0 CONCLUSION

An analysis has been conducted on a single-item production model with time-dependent production and demand rates. Numerical results for various performance measures were obtained using the fourth-order Runge–Kutta method implemented in MATLAB R2016a. The proposed model can be extended to different server configurations, providing a more generalized framework under time-dependent conditions and making the model more realistic and applicable to practical scenarios.

REFERENCES

Ataullah, M. (2022). *Time-Dependent Production Inventory Model with Random Switching Time for Non-Perishable Items*. CodexCL. Retrieved from https://media.codexcl.com/documents/1_Ekramol_Printable_Edit.pdf

Ataullah, M., & Islam, M. E. (2020). *Stochastic Inventory System With Different Rates of Production*. International Journal of Operations Research and Information Systems, 11(2), 1–14. https://doi.org/10.4018/IJORIS.2020040103

Ataullah, M., & Islam, M. E. (2020). *Transient Analysis of Production Inventory System with Different Rates of Production and Random Switching Time*. International Journal of Advanced Science and Research, 6(1), 52–58. Retrieved from https://ijasre.net/index.php/ijasre/article/view/52

Ataullah, M., & Islam, M. E. (2020). *Transient Analysis of Production Inventory System with Different Rates of Production and Random Switching Time*. Academia.edu. Retrieved from https://www.academia.edu/38517567*Mohammad Ekramol Islam, et al. (2019). Transient Analysis of Production Inventory System with Different Rates of Production and Random Switching Time* – (paper providing direct transient results for a two-rate model with random switching). ijasre.net

Ben-Daya, M., et al. (2008). *Economic production quantity model with a shifting production rate* (EPQ with random shifting time). Wiley Online Library

Emunds, T., & Nießen, S. (2024). *Utilizing Phase-Type Distributions for Queueing-Based Railway Junction Performance Determination*. arXiv. https://doi.org/10.48550/arXiv.2412.04188

Krishnamoorthy, A., et al. (2013). *Stochastic decomposition in production inventory with random processing time* (matrix-analytic and level-dependent approaches). ScienceDirect

Mohebbi, E., et al. (2006). *A production-inventory model with randomly changing parameters / limited storage* (analysis with compound Poisson demand and switching). ScienceDirect

Poormoaied, S., & Demirci, E. Z. (2021). *Analysis of an Inventory System with Emergency Ordering Option at the Time of Supply Disruption*. OR Spectrum, 43(4), 1007–1045. https://doi.org/10.1007/s00291-021-00636-x

Wang, X., Jia, Z., & Dai, Y. (2023). *Transient Analysis of Production Performance and Energy Consumption in Geometric Flexible Production Systems*. ScienceDirect. https://doi.org/10.1016/j.eswa.2023.123065

Wang, X., Jia, Z., & Dai, Y. (2024). *Transient Analysis of Production Performance and Energy Consumption in Geometric Flexible Production Systems*. Expert Systems with Applications, 245, 123065. https://doi.org/10.1016/j.eswa.2023.123065